MUSICAL-MATHEMATICAL MODEL

DEVELOPMENT OF A MATHEMATICAL MUSICAL TEMPO MODEL FOR THE MID SEGMENT OF THE SECTION "MATHEMATICS & ARTS II" AT THE 2ND MOVEMENT OF "THE DREAMS CONCERTO": "EXACT MODEL OF 7!".

Raimundo Rodulfo

Initial Criteria:

Starting concept: 3 parallel progressions P_i between t_0 y t_f. **i**: progression index. $i = \{1,..,3\}$

2 levels of tempo superposition:

Level 1: Progressions Superposition. $PS_1(t) = \{P_1,..,P_j\}$

Level 2: Intrinsic Superposition on each P_i. $PS_2(i,t) = \{T_{i,1}, T_{i,2}, ...\}$

Gradual superposition of tempos on each layer:
$t_0 (P_i) < t_0 (P_{i+1})$; Level 1, $i = \{1,..,2\}$. Each progression is related to an instrumental section.
$t_0 (T_{i,j}) < t_0 (T_{i,j+1})$; Level 2, $j = \{1,..,\#PS_2(i,t_f)\}$. Each tempo is related to an instrument.

$\forall$ **i,j** is true that: $t_f (P_i) = t_f (T_{i,j}) = t_f$. Tempo synchronization with one common ending beat is enforced, common to all the progressions and all the tempos.

The Intrinsic Superposition Density (ISD) is ruled by a binary descendent pattern:
$ISD (P_i) = 2^{3-i}$, where $ISD (P_i) = \#_{max}PS_2(i,t) = \#PS_2(i,t_f)$

Then, we obtain the superposition sequences:

Level 1: $\{ P_1 \} \rightarrow \{ P_1 , P_2 \} \rightarrow \{ P_1 , P_2 , P_3 \}$

Level 2:

$\{ T_{1,1} \} \rightarrow \{ T_{1,1} , T_{1,2} \} \rightarrow \{ T_{1,1} , T_{1,2} , T_{1,3} \} \rightarrow \{ T_{1,1} , T_{1,2} , T_{1,3} , T_{1,4} \}$; P_1

$\{ T_{2,1} \} \rightarrow \{ T_{2,1} , T_{2,2} \}$; P_2

$\{ T_{3,1} \}$; P_3

By introducing musical criteria on the analysis, on Level 1 the instrumental sections will be assigned as following:

P1: Percussion, **P2:** Harmonic Base, **P3:** Soloist

This way, the percussion element is granted in all the progressions, and the harmonic base element is granted in the soloist instrument progression.
Same way is assigned at Level 2:

$T_{1,1}$ **:** Drums 1, $T_{1,2}$ **:** Percussion 1, $T_{1,3}$ **:** Drums 2**,** $T_{1,4}$ **:** Percussion 2

$T_{2,1}$ **:** Bass **,** $T_{2,2}$ **:** Keyboard

$T_{3,1}$ **:** Electric Guitar

The model will be split in as many segments as dominant tempos it has. The tempos are generated from 7! : $T \in \{1,..,7\}$. T=1 is implicit in all, T=2 in T=4, and T=3 in T=6. This way we get 4 primary tempos: $T = \{4,..,7\}$

To avoid the intrinsic tempo repetition on the progressions:
$\forall$ **i,j,k** is true that: $T_{i,j} \neq T_{i,k}$, $j \neq k$ **(I)**
This is possible with a linear tempo model $T_{i,j+1} = T_{i,j} \pm 1$

To grant the progressive appearance of tempos:
$\forall$ **t** on $\mathbf{t_0 \leq t \leq t_f}$ is true that $\mathbf{PS_2(i,t) \in PS_2(1,t)}$ **(<u>II</u>)**

To grant the contraposition of tempos, let's utilize alternating order on the tempo appearance at each new progression. That way:

$\mathbf{T_{i,j+1} = T_{i,j} + (-1)^{i-1}}$ **(<u>1</u>)** , and per patterns symmetry: $\min_{j=1}^{ISD(P_i)} \{T_{i,j}\} = 4$

Per musical considerations, $\mathbf{P_1}$ and $\mathbf{P_2}$ (rhythmic and harmonic base elements) will utilize constant tempo sequences, and $\mathbf{P_3}$ (soloist element) will follow a sequential pattern of variable tempo, where all the model tempos will be progressively recreated with Iteration Grade **I** constant. Therefore:
$\mathbf{T_{1,1} = 4 \; ; \; T_{1,2} = 5 \; ; \; T_{1,3} = 6 \; ; \; T_{1,4} = 7 \; ; \; T_{2,1} = 5 \; ; \; T_{2,2} = 4 \; ; \; T_{3,1} = f(t) = 4I \rightarrow 5I \rightarrow 6I \rightarrow 7I}$

<u>1st Progression Analysis.</u>

$\mathbf{P_1}$ by being the only progression that superposes all the dominant tempos of the model will determine its segmentation. So are defined 4 segments **S**:

$S_k = \bigcap_{j=1}^{k} T_{1,j} - \bigcap_{j=1}^{k+1} T_{1,j}$; then: $\mathbf{t_0\,(S_k) = t_0\,\{T_{1,1}, .., T_{1,k}\}}$ **y** $\mathbf{t_f\,(S_k) = t_0\,\{T_{1,1}, .., T_{1,k+1}\}}$

Dominant Tempo of segment $\mathbf{S_i}$: $\mathbf{T_{1,i} = i + 3 \;\; ; \; i = \{1,..,4\}}$

Per musical criteria, time will be measured by primary discrete units to be called **TPU** (Time Primary Unit), equivalent to 1 sixteenth note.

Timing Pattern Synchronism Condition: We will be define quantic units for the tempo superposition groups, so that:

$$Q(S_i) = \underset{j=1}{\overset{i}{mcm}}(T_{1,j}) = mcm\{4,..,3+i\} \quad \text{[TPU/Quantum]}$$

A quantic iteration descendent binary pattern will be followed, it is to say:
$\mathbf{I_Q(S_i) = 2^{4-i}}$ [Quanta]; so, the quantity of TPU's per segment is given by:

$$U(S_i) = I_Q(S_i)Q(S_i) = 2^{4-i} \underset{j=4}{\overset{3+i}{mcm}}(j) \quad \text{[TPU]} \qquad \textbf{(\underline{2})}$$

The duration in seconds of each segment is obtained converting TPU to sec.
In this case, 5 TPU $\Leftrightarrow$ 1 sec. Then:

$$D(S_i) = \frac{1}{5} 2^{4-i} \underset{j=4}{\overset{3+i}{mcm}}(j) \quad \text{[sec]} \qquad \textbf{(\underline{3})}. \text{ This way:}$$

$$t_0(S_i) = \sum_{k=1}^{i-1} U(S_k) = \sum_{k=1}^{i-1}\left(2^{4-k} \underset{j=4}{\overset{3+k}{mcm}}(j)\right) \quad \text{[TPU]} \qquad \textbf{(\underline{4})} \quad \text{and} \quad \mathbf{t_f\,(S_i) = t_0\,(S_{i+1})} \qquad \textbf{(\underline{5})}$$

From **(<u>4</u>)** y **(<u>5</u>)**: $\mathbf{t_0\,(S_1)} =$ **<u>0</u>** ; $\mathbf{t_f\,(S_1) = t_0\,(S_2)} =$ **<u>32</u>** ; $\mathbf{t_f\,(S_2) = t_0\,(S_3)} =$ **<u>112</u>** ; $\mathbf{t_f\,(S_3) = t_0\,(S_4)} =$ **<u>232</u>**

The following tempo patterns are defined:

Primary Pattern: The 4 primary tempos obtained from 7! : **T = {4,..,7}**. The starting of each primary tempo pattern $\mathbf{T_{1,i} = i+3}$ occurs at $\mathbf{t_0}$ **(**$\mathbf{S_i}$**)**, in the progressive factorial superposition from 4 to 7.

Secondary Pattern: Defined for each primary tempo, from a fixed number of iterations. Per musical reasons, an octal iteration pattern is chosen. We obtain then **PT = 8T.** The structure of the octal pattern is decomposed the following way:

$\mathbf{PT_{1,i} = 8T_{1,i} = 8(i + 3) = 4(i + 3)}$ linear $\rightarrow$ **4 (i + 3)** out of phase. **(III)**

The relative position of the linear accent, and the out of phase accent are random functions in the range **{1,.., i + 3}**

Total values of the model, calculated from $\mathbf{P_1}$:

From **(2)**: $UT = \sum_{k=1}^{4}\left(2^{4-i}\ \underset{j=4}{\overset{3+i}{mcm}}(j)\right)$ **= 652 TPU**

From **(3)**: $DT = \frac{1}{5}\sum_{i=1}^{4}\left(2^{4-i}\ \underset{j=4}{\overset{3+i}{mcm}}(j)\right)$ **= 130,4 sec.**

2nd Progression Analysis.

From **(1)** is obtained that the second tempo sequence of segment $\mathbf{S_i}$ is opposed to the first, so that the occurrence of their dominant tempos is given by: $\mathbf{T_{2,i-2} = 8 - i}$

From criterion **(II)** we know that in this case, **i = {3,4}**

To guarantee the synchronism in the patterns superposition of equal tempos from different progressions, the structure of octal patterns explained at **(III)** enforces that the starting of tempos in $\mathbf{P_2}$ occur this way:

$$\mathbf{t_0\,(T_{2,i-2}) = t_0\,(S_{5-i}) + PT_{2,i-2}}\left\{Int\left[\frac{t_0(Si)-t_0(S5-i)}{PT2,i\text{-}2}\right]+1\right\}\ ;\quad \mathbf{i=\{3,4\}}\ \Rightarrow$$

$$\mathbf{t_0(T_{2,i-2})}=\sum_{j=1}^{4-i}\left(2^{4-j}\ \underset{k=4}{\overset{3+j}{mcm}}(k)\right)+8(8-i)\left(Int\left\{\frac{1}{8-i}\left[\sum_{j=1}^{i-1}\left(\frac{1}{2^{j-1}}\underset{k=4}{\overset{3+j}{mcm}}(k)\right)-\sum_{j=1}^{4-i}\left(\frac{1}{2^{j-1}}\underset{k=4}{\overset{3+j}{mcm}}(k)\right)\right]\right\}+1\right);\ \mathbf{(6)}$$

For the Bass: $\mathbf{t_0\,(T_{2,1}) = 32 + 40\,\{\,Int\,[0{,}2\,(4 + 10 - 4)] + 1\,\} = 32 + 40\,(2 + 1) = \underline{152}}$

For the Keyboard: $\mathbf{t_0\,(T_{2,2}) = 0 + 32\,\{\,Int\,[0{,}25\,(4 + 10 + 15 - 0)] + 1\,\} = 32\,(7 + 1) = \underline{256}}$

3rd Progression Analysis.

The behavior of $\mathbf{T_{1,3}}$ as variable in function of time, as explained before, presents the soloist instrument (electric guitar) describing a ascendant linear series of the 4 dominant tempos of the model $\mathbf{(T_{1,1}\,,..,\,T_{1,4})}$, in sequences of iteration **I** constant. As $\mathbf{t_0\,(P_2) = t_0\,(T_{2,1}) > t_0\,(S_3)}$ - see **(III)** and **(6)**- then in order to guarantee the non-excluding compliance with the superposition sequence of Level 1, and the contraposition criterion of **(1)**, we make: $\mathbf{t_0\,(P_3) > t_0\,(S_4)}$

From equation $\mathbf{T_{3,1} = f(t) = 4I \rightarrow 5I \rightarrow 6I \rightarrow 7I}$ we know that at $\mathbf{t_0\,(T_{3,1})^{+}}$, $\mathbf{T_{3,1} = 4}$. Then, its beginning coincides with the beginning of $\mathbf{T_{2,2}}$. Therefore: $\mathbf{t_0\,(P_3) = t_0\,(T_{3,1}) = t_0\,(T_{2,2}) = \underline{256}}$ $\Rightarrow$

$$\mathbf{4I + 5I + 6I + 7I = t_f - t_0\,(P_3) = 652 - 256 = 396\ TPU} \Rightarrow \mathbf{I} = \frac{396}{22} = \mathbf{\underline{18}}$$

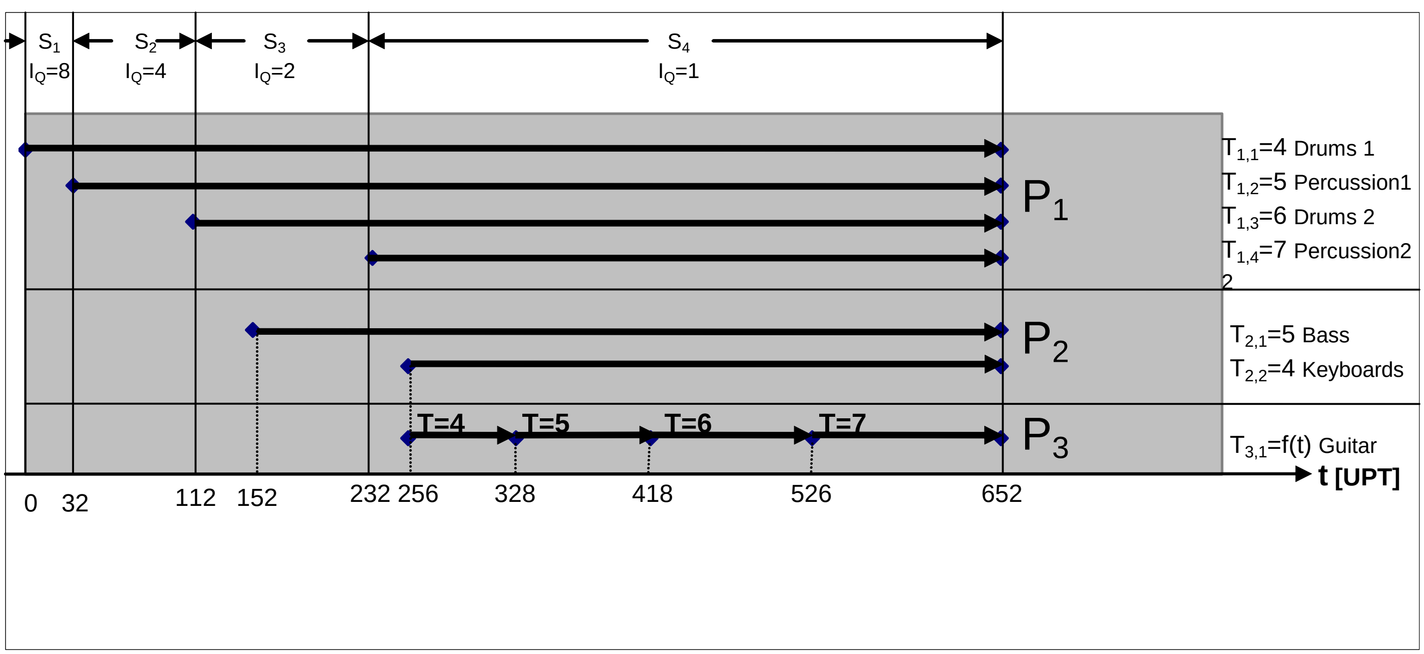
So, $\mathbf{P_3}$ must be structured with ascendant linear patterns of **18** iterations.

From the analysis and calculations done above, are obtained the following charts:

Table 1. Values of the model.

S: Segment		1	2	3	4	**Iterations per tempo**	**TPU**	**Octal Pattern**	**Starting in P_1**	**Starting in P_2**	**Starting in P_3**
T: Tempo	**$T_{i,1}$**	4	4	4	4	163	652	32	0	256	256
	$T_{i,2}$		5	5	5	124	620	40	32	152	328
	$T_{i,3}$			6	6	90	540	48	112		418
	$T_{i,4}$				7	60	420	56	232		526
Q: Quantum		4	20	60	420						
I_Q: Quantic Iterations		8	4	2	1						
U_0: (Beginning)		0	32	112	232						
U_f: (Ending)		32	112	232	652	**Total**					
U: TPU (U_f - U_0)		32	80	120	420	**652**					
D: Duration (sec)		6.4	16	24	84	**130.4**	**=2'10"4**				

Diagram 1. Time Diagram



GEOMETRIC REPRESENTATION OF THE MODEL

In order to obtain a graphic representation of the model, let's start from its fundamental equations, and from some principles of time-space analogy that allows us to relate the rhythmic patterns modeled initially with bi-dimensional shapes. Let's start so from a linear chromatic representation, where it's shown the intrinsic and extrinsic superposition of the 3 progressions of the model by a spatial linear superposition of chromatic patterns, and let's continue then with the development of a geometric representation juxtaposed to the first one. On each one of the two layers, we will study the progressions 1 and 2 as one only case, because both are rhythmic parallel arrays with intrinsic superposition, and let's study progression 3 as a separate case, because despite the others, it is a rhythmic series array without intrinsic superposition.

Analysis of the 1st layer of the graphic model: Linear Chromatic Representation

Time will be represented linearly, assigning primary unitary increments equivalent to 1 TPU, correspondent in the musical representation to 1 sixteenth note; this way, the representation will be taken to a total scale of 652 units in its time-associated dimension. The patterns will be represented proportionally over the horizontal axis, and the segmentation will implicitly exist in the starting positions of the superposed frames. The differentiation between musical instruments will be represented by assigning an unique color $\mathbf{C_{i,j}}$ to each one, determined by the RGB code obtained directly from the starting position of each quantic iteration (see the time diagram of the model). This way we take time to the frequency domain, utilizing a digital function:

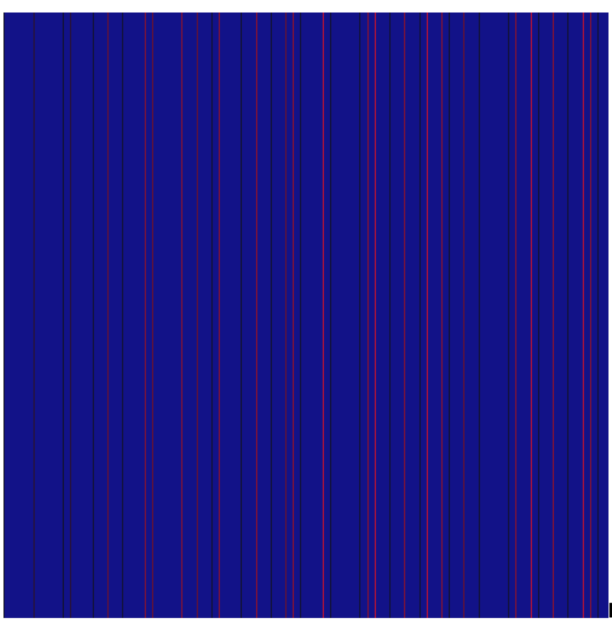
RGB ($C_{i,j}$) = f (t_0 ($T_{i,j}$)) = t_0 ($T_{i,j}$) ; (<u>7</u>)

The calculation of each $\mathbf{t_0}$ **($T_{i,j}$)** is made from the fundamental equations deducted on the analysis of the three progressions. To represent the two levels of superposition and the tempo superposition density on the dimensional scale, we will choose the starting beats of each tempo pattern **PT ($T_{i,j}$)** as events and they will be visualized with chromatic impulses **Sa (PT ($T_{i,j}$),$C_{i,j}$)**. That way, the impulses will occur in the following positions:

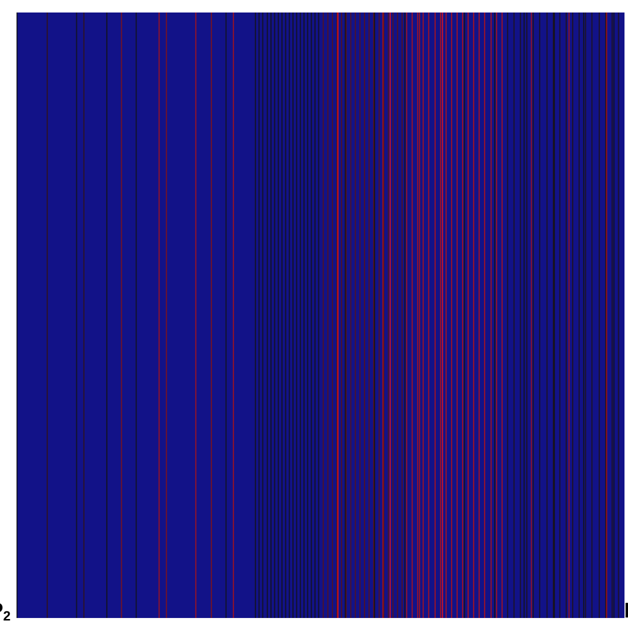
x **(Sa (PT ($T_{i,j}$), $C_{i,j}$) = t_0 ($T_{i,j}$) + k $PT_{i,j}$; k = {0,..,I} (<u>8</u>)**

where $PT_{i,j} = \begin{cases} 8T_{i,j} & ; i=1,2 \\ T_{i,j} & ; i=3 \end{cases}$, and $I = \begin{cases} \dfrac{652 - t_0(T_{i,j})}{PT_{i,j}} & ; i=1,2 \\ 18 & ; i=3 \end{cases}$

That way we obtain the following development of linear chromatic progressions:



P_1,P_2



P_1,P_2,P_3

Analysis of the second layer of the graphic model: Geometric Representation

In a similar way as in the previous layer, the differentiation between tempos (or musical instruments) will be represented by assigning an unique color $\mathbf{C_{i,j}}$ to each instrument, which will be obtained in this case from a linear function of the beginning position (or tempo) of each pattern (see the Time Diagram of the model). That way we have:

$$\mathbf{RGB\ (C_{i,j}) = \mathit{f}_2\ (t_0\ (T_{i,j})) = A\ t_0\ (T_{i,j}) + B} \quad ; \quad \mathbf{(\underline{9})}$$

Where **A** and **B** are constants to be determined, and the value $\mathbf{C_{i,j}}$ obtained will be converted to digital format RGB, on which every color is represented as a hexadecimal integer number, motive why we will round **A** to the next closer integer. The calculation of each $\mathbf{t_0\ (T_{i,j})}$ is made from the fundamental equations deducted on the three progressions analysis. Starting from the color-position analogy, we assume that the position **0** has to be assigned to the absence of color (black). This way, we state that:

$$\mathbf{C\ (0) = RGB\ (black) = H0} \quad \Rightarrow \quad \mathbf{\underline{B = 0}}$$

If we consider that in the final instant of the model it reaches its maximum tempo superposition density (or colors), we got that:

$$\mathbf{C\ (652) = RGB\ (white) = H00FFFFFF} \quad \Rightarrow \quad \mathbf{\underline{A = 25732}\ (decimal)}$$

The principles on which is founded the mathematical model: the 2 levels of tempo superposition, the gradual tempo superposition on each level, the tempo synchronicity with an only ending beat common to all the progressions and all the tempos, the intrinsic superposition density controlled by a descendent binary pattern, the progressive appearance of opposed tempos, the segmentation, the symmetry and the tempo patterns synchronicity; all will be recreated on the geometric representation by polygons emulating in each one of their sides one tempo unit in the correspondent rhythmic pattern.

For $\mathbf{P_1}$ and $\mathbf{P_2}$, it will be represented the exact superposition of synchronized tempos with the exact concentric polygonal contention; in this case, for example, the synchronic superposition of quantic patterns of 6 in patterns of 5 is going to be represented as a hexagon inscribed inside a pentagon. The analysis of a geometric relation of this kind is shown on Figure 1, where a $\mathbf{N_1}$ sides polygon exactly contains a second $\mathbf{N_2}$ sides polygon. The correspondent lateral lengths are $\mathbf{L_1}$ and $\mathbf{L}_2$, the radial distances orthogonal to the sides are $\mathbf{D_1}$ and $\mathbf{D_2}$, and each one is exactly contained by a circumference of radius $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively. The angles that define each side are equal to $\mathbf{2\pi/N_1}$ and $\mathbf{2\pi/N_2}$ respectively, therefore:

$$\theta_1 = \pi/N_1 \quad \text{and} \quad \theta_2 = \pi/N_2 \quad ; \quad \mathbf{(\underline{10})}$$

So, we obtain from the exterior polygon the radius of the exterior circumference of the interior polygon:

$$\mathbf{D_1 = R_2} \quad \Rightarrow \quad \mathbf{R_1 \cos(\theta_1) = R_2} \quad \Rightarrow \quad \text{from } \mathbf{(\underline{10})}: \quad \mathbf{R_2 = R_1 \cos(\pi/N_1)} \quad ; \quad \mathbf{(\underline{11})}$$

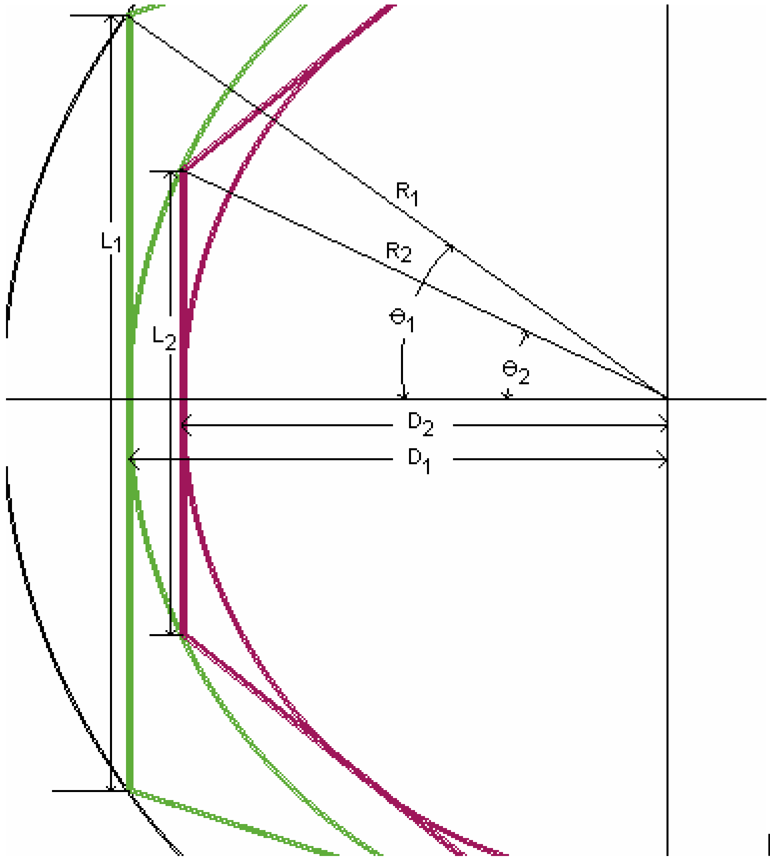


Figure 1

The first generated polygon corresponds to $\mathbf{T_{1,1} = 4}$ at $\mathbf{t_0\ (T_{1,1}) = 0}$, therefore it is a square with **L = 652** and imaginary exterior circumference of $R1{,}1 = 652\sqrt{2}$.

This first exterior circumference is not seen in the graphics, because exceeds its maximum dimensions of 652 TPU. The circumference inscribed in this first polygon determines the exterior radius of the next polygon corresponding to $\mathbf{T_{1,2} = 5}$ at $\mathbf{t_0\ (T_{1,2}) = 32}$; then, $\mathbf{R_{1,2} = 652}$. As the polygons must represent the pattern superposition in the order they are appearing, we will assume as a convention the indexation given by the following tempo sequence, sorted by beginning time $\mathbf{t_0}$ (see Diagram 1):

$$\mathbf{N_1 \rightarrow \ldots \rightarrow N_6 \;=\; T_{1,1} \rightarrow T_{1,2} \rightarrow T_{1,3} \rightarrow T_{2,1} \rightarrow T_{1,4} \rightarrow T_{2,2} \;=\; 4 \rightarrow 5 \rightarrow 6 \rightarrow 5 \rightarrow 7 \rightarrow 4}$$

From the first polygon ($\mathbf{R_1 = 652}$) is deducted the following fundamental equation:

$$\mathbf{R_i = R_{i-1} \cos(\pi / N_{i-1}) \quad ; \quad i = \{2,..,6\}} \qquad \textbf{(12)}$$

The coordinates ***(x,y)*** of the common centre for the 6 polygons of $\mathbf{\{P_1 \cup P_2\}}$ are **(326,326)**, given the maximum pre-established area of **652 x 652** for the representation of the model. Once it is known the radius of each polygon exterior circumference, we need an algorithm that allows us to draw each one of its sides, that is a set of $\mathbf{N_i}$ lines defined by the coordinates of beginning and ending $\mathbf{(x_{i,j,0},y_{i,j,0}),(x_{i,j,f},y_{i,j,f})}$, where $\mathbf{i = \{2,..,6\}}$ and $\mathbf{j = \{0,..,N_i\text{-}1\}}$. Starting from a first vortex over axis X, we get the fundamental equations for any phase angle ϕ_i:

$$\mathbf{(x_{i,j,0},y_{i,j,0}) = (\,326 + R_i \cos(\phi_i + 2j\pi/N_i)\,,\, 326 + R_i\, \mathrm{sen}(\phi_i + 2j\pi/N_i)\,)} \quad \text{and}$$

$$\mathbf{(x_{i,j,f},y_{i,j,f}) = (\,326 + R_i \cos[\phi_i + 2(j+1)\pi/N_i]\,,\, 326 + R_i\, \mathrm{sen}[\phi_i + 2(j+1)\pi/N_i]\,)} \quad ; \quad \textbf{(13)}$$

where $\mathbf{i = \{2,..,6\}\ ,\ j = \{0,..,N_i\text{-}1\}}$

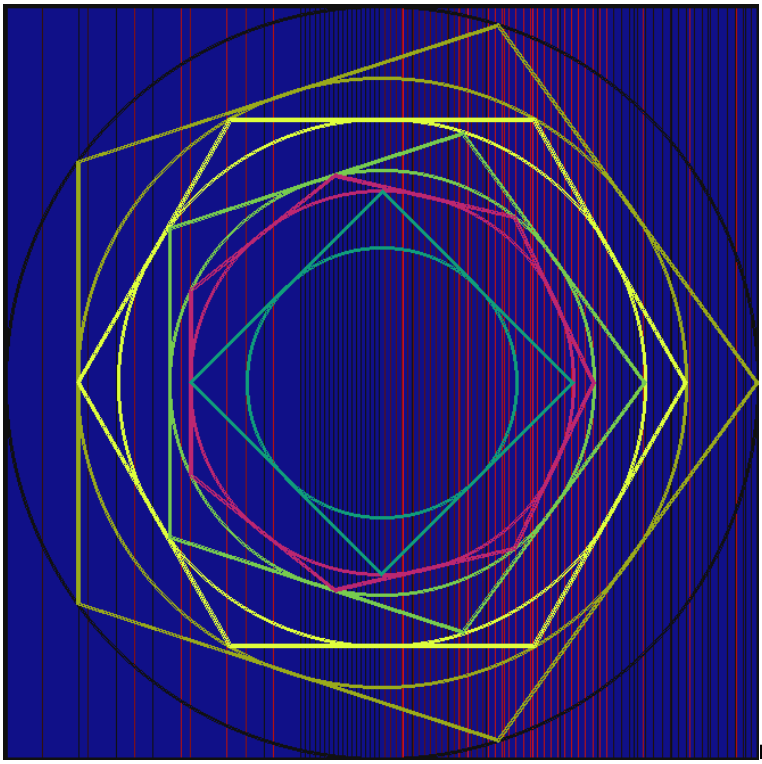
Substituting **(12)** in **(13)**, we get the fundamental equations for the algorithm of geometric modeling of $\mathbf{\{P_1 \cup P_2\}}$:

$(x_{i,j,0},y_{i,j,0}) = (326 + R_{i-1} \cos(\pi/N_{i-1}) \cos(\phi_i + 2j\pi/N_i) , 326 + R_{i-1} \cos(\pi/N_{i-1}) \,\mathrm{sen}(\phi_i + 2j\pi/N_i))$;

$(x_{i,j,f},y_{i,j,f}) = (326 + R_{i-1} \cos(\pi/N_{i-1}) \cos[\phi_i + 2(j+1)\pi/N_i] , 326 + R_{i-1} \cos(\pi/N_{i-1})\mathrm{sen}[\phi_i + 2(j+1)\pi/N_i])$

where $i = \{2,..,6\}$, $j = \{0,..,N_i-1\}$, $R_1 = 326$, $N_1 \rightarrow \ldots \rightarrow N_6 = 4 \rightarrow 5 \rightarrow 6 \rightarrow 5 \rightarrow 7 \rightarrow 4$; **(14)**

That way is generated the following representation with null phase juxtaposed to the first layer:



P_1,P_2

For P_3, it is going to be represented the exact superposition of the series of 4 tempos $T_{3,1} \rightarrow \ldots \rightarrow T_{3,4}$ synchronized with the frame of $T_{2,2} = 4$, inscribing one series of 4 polygons of adjacent sides inside the interior circumference of the polygon of $T_{2,2}$ (see the interior square of the model of P_1,P_2 shown above). An arrangement that complies with this premise is shown in Figure 2. The common axis to the pentagon, the hexagon and the heptagon, correspondent by analogy to $T_{3,2}$, $T_{3,3}$ and $T_{3,4}$, must be equal to the interior diameter of $T_{2,2}$, which is defined by the radius of the equation **(12)** in the following iteration, when $i = 7$.

From **(12)** with $i = 7$: $D = 652 \cos^2(\pi/5) \cos(\pi/6) \cos(\pi/7) \cos(\pi/4)$; **(15)**

From analysis of Figure 2, we infer the following geometric relations:

$L = 2R_i \,\mathrm{sen}(\pi/i) = \sqrt{2} R_4$; $i = \{5,..,7\} \Rightarrow \sqrt{2} R_4 = R_5 \,\mathrm{sen}(\pi/5) = R_6 \,\mathrm{sen}(\pi/6) = R_7 \,\mathrm{sen}(\pi/7)$; **(16)**

Expressing all in function of R_5:

$$R_6 = R_5 \frac{\mathrm{sen}\left(\frac{\pi}{5}\right)}{\mathrm{sen}\left(\frac{\pi}{6}\right)} \quad , \quad R_7 = R_5 \frac{\mathrm{sen}\left(\frac{\pi}{5}\right)}{\mathrm{sen}\left(\frac{\pi}{7}\right)} \quad \text{and} \quad R_4 = \frac{1}{\sqrt{2}} R_5 \,\mathrm{sen}\left(\frac{\pi}{5}\right) \; ; \quad \textbf{(17)}$$

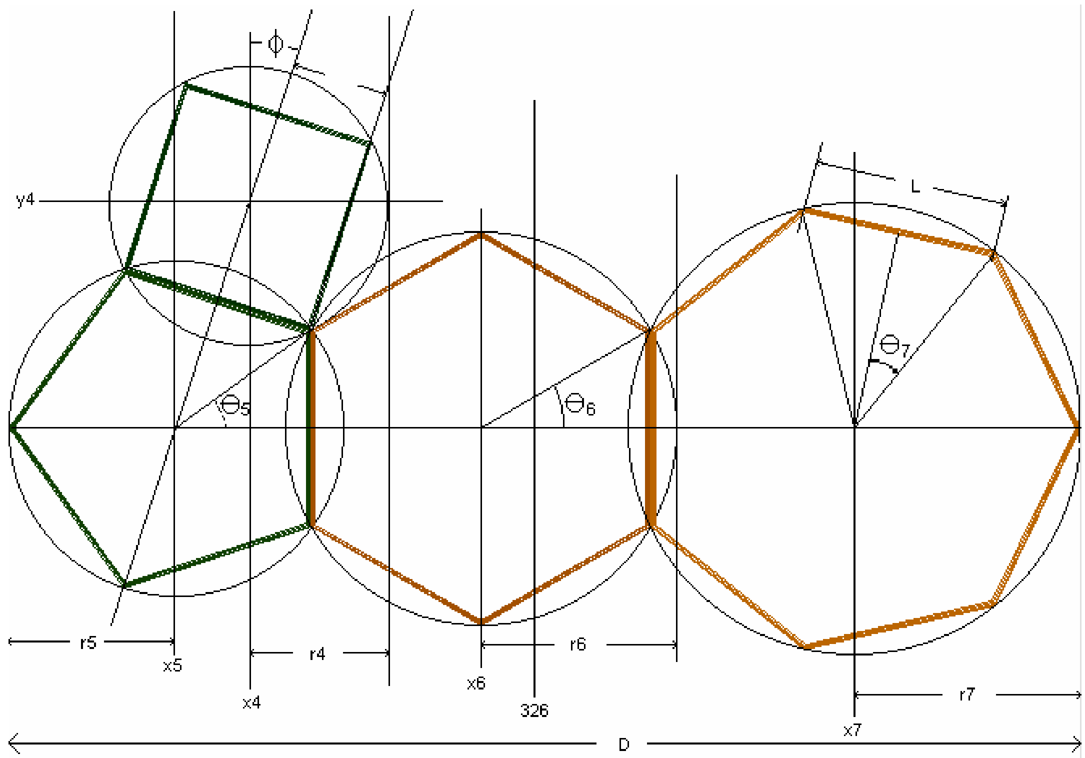


Figure 2.

To draw the polygons we just need to get the coordinates **(x,y)** of the centers of their exterior circumferences. From Figure 2 we get:

$(x_5,y_5) = (326 - \frac{1}{2}D + R_5\ ,\ 326)$, and from this first value, the algorithm to calculate the centers for the hexagon and the heptagon:

$(x_i,y_i) = (x_{i-1} + R_{i-1}\cos[\pi / (i-1)] + R_i\cos(\pi / i)\ ,\ 326)$; $i = \{6,7\}$; **(18)**

In the case of the quadrilateral correspondent to $T_{3,1} = 4$, we get the phase angle ϕ and the coordinates from the geometrics relations with the pentagon. That way:

$\phi = \pi/2 - 2\theta_5 = \pi/2 - 2\pi/5 = \underline{\pi/10}$

The distance from the center of the quadrilateral to the center of the pentagon is the sum of their respective orthogonal radial distances, and its angle with the horizontal is $2\theta_5 = 2\pi/5$; then:

$x_4 = R_5 + R_5[\cos(\pi/5) + \mathrm{sen}(\pi/5)]\cos(2\pi/5)$ and

$y_4 = 326 + R_5[\cos(\pi/5) + \mathrm{sen}(\pi/5)]\,\mathrm{sen}(2\pi/5)$; **(19)**

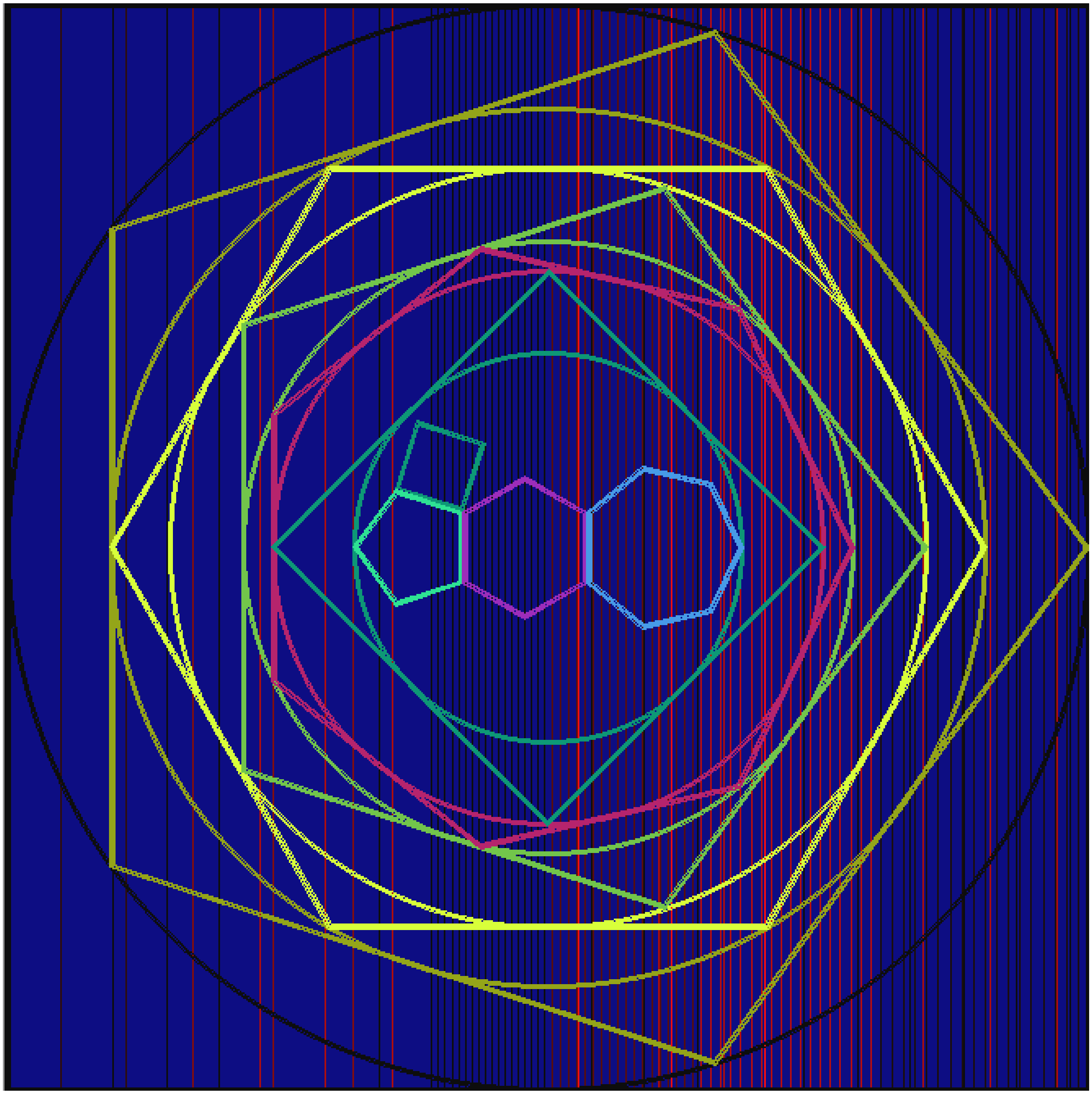
Finally, let's calculate R_5 from the known distance **D**, and the geometric relations between the polygons at the horizontal axis (see Figure 2):

$R_5[1 + \cos(\pi/5)] + 2R_6\cos(\pi/6) + R_7[1 + \cos(\pi/7)] = D$; **(20)**

Substituting **(15)** and **(17)** in **(20)** :

$$R5 = \frac{652\cos^2\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{5}\right)+2\frac{\mathrm{sen}\left(\frac{\pi}{5}\right)}{\mathrm{tg}\left(\frac{\pi}{6}\right)}+\frac{\mathrm{sen}\left(\frac{\pi}{5}\right)}{\mathrm{sen}\left(\frac{\pi}{7}\right)}\left[1+\cos\left(\frac{\pi}{7}\right)\right]} \quad ; \quad \textbf{(21)}$$

That way, is obtained the final representation of the model in two layers:



This model has been computer-generated, developing software that utilizes algorithms based on the equations deducted from the prior analysis.